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Application of Volume Balances and the Differential Diffusion Equation to Filtration

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ABSTRACT

A mathematical model describing the formation of filter cakes in filtration is presented. Filtration equations are derived from volume balances and Darcy's equation, and they are presented in the same form as partial differential diffusion equations. The general form of the model is applicable to three-dimensional formation of a filter cake. Both compressible and incompressible cakes are considered. A calculated example for incompressible cake filtration is presented.

INTRODUCTION

In cake filtration, solids are separated from liquid with a filter medium. Liquid flows through the medium while solid particles are retained on the surface of the medium, forming a cake. A cake is termed incompressible when the porosity, i.e., void fraction of cake, and specific cake resistance are constant throughout the cake (1). In a compressible cake, porosity usually decreases toward the filter medium.

Many studies on the mathematical modeling of compressible cake filtration have been presented, e.g., Chase and Willis (2), Stamatakis and Chi Tien (3), Tiller et al. (4), Wakeman (5), Smiles (6), and Shirato et al. (7). The filtration of compressible fiber suspensions has also been studied with the same basic principles, e.g., Ingmanson (8), Kovasin et al. (9), and Nordén et al. (10).

Recently Stamatakis and Chi Tien (3) derived equations describing the formation and growth of filter cakes. Their equations are based on the general discussion on filtration presented by Tiller et al. (4). Stamatakis and Chi Tien (3) also presented a method for the numerical solution of these equations. Only one-dimensional filtration cases are considered in the study, and the compressible cake is assumed to be finite.

Smiles (6, 11) proposed an approach to filtration where the filtration equations are presented in the form of differential diffusion equations. This approach is applicable to one-dimensional cases. Smiles employs a material coordinate based on the distribution of the solid in his approach, and he uses the moisture ratio, i.e., the specific volume of water per specific volume of solid, as the dependent variable. However, as pointed out by Wakeman (5) and Tosun (12), the approach is conceptually difficult and ignores accepted filtration terminology.

In this study on cake filtration, equations of continuity are based on volume coordinates, and volume average velocity is used as average velocity. The pressure loss of liquid in cake is described by Darcy's law, and liquid flow is assumed to be laminar. A force balance is used to correlate compressive pressure with hydraulic pressure, and the consistency of solids is related to compressive pressure with a constitutive function. These equations are combined, and an equation for cake filtration is presented. This equation has the same form as the partial differential equation in diffusion with a nonconstant volumetric diffusivity. The equations are presented in three-dimensional form.

VOLUME BALANCES

In the literature, no systematic approach to volume balances has been found apart from Nordén (13), although in some studies volume balances have been used in some simple form, i.e., Renault and Wallender (14), Core and Mulligan (15), and Rohani and Baldyga (16).

Volume balances may be derived from the corresponding mass balances (13). The differential component and total mass balances are, respectively,

$$r_i = \frac{Dc_i}{Dt} + c_i \nabla \cdot \mathbf{w} + \nabla \cdot \mathbf{j}_i \tag{1}$$

$$r = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{w} + \sum_{i=1}^{I} \nabla \cdot \mathbf{j}_i \tag{2}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla$$

The velocity $\mathbf{w}$ can be mass, molar, or volume-average velocity.

The substitutions of $\dot{V}'''_{\rm gen}$, $\dot{V}'''_{i,\rm gen}$, $\rho v = 1$, $c_i\bar{v}_i$, and $\mathbf{j}_i\bar{v}_i$ for r, r_i, ρ, c_i, and $\mathbf{j}$, respectively, gives the differential component and total volume balances (13).

$$\dot{V}'''_{i,\rm gen} = \frac{D(c_i\bar{v}_i)}{Dt} + (c_i\bar{v}_i)\nabla\cdot\mathbf{w} + \nabla\cdot(\mathbf{j}_i\bar{v}_i) \tag{3}$$

$$\dot{V}'''_{\rm gen} = \nabla\cdot\mathbf{w} + \sum_{i=1}^{I} \nabla\cdot(\mathbf{j}_i\bar{v}_i) \tag{4}$$

In this study, equations describing the three-dimensional two-phase flow of slurry in filtration are derived from the preceding volume balances.

The total volume balance takes the form

$$\dot{V}'''_{\rm gen} = \nabla\cdot\mathbf{w} + \nabla\cdot(\mathbf{j}_{\rm f}\bar{v}_{\rm f} + \mathbf{j}_{\rm s}\bar{v}_{\rm s}) \tag{5}$$

The solids volume balance is

$$\dot{V}'''_{\rm s,gen} = \frac{D(c\bar{v}_{\rm s})}{Dt} + (c\bar{v}_{\rm s})\nabla\cdot\mathbf{w} + \nabla\cdot(\mathbf{j}_{\rm s}\bar{v}_{\rm s}) \tag{6}$$

And a similar balance for filtrate or liquid is

$$\dot{V}'''_{\rm f,gen} = \frac{D(c_{\rm f}\bar{v}_{\rm f})}{Dt} + (c_{\rm f}\bar{v}_{\rm f})\nabla\cdot\mathbf{w} + \nabla\cdot(\mathbf{j}_{\rm f}\bar{v}_{\rm f}) \tag{7}$$

With the following assumptions:

no reactions or dissolving of solids occur.
partial specific volumes $\bar{v}_{\rm s}$ and $\bar{v}_{\rm f}$ are constant.

Equations (6) and (7) reduce to the corresponding mass balances.

As there is no generation of mass in the system, it is seen from the mass balances that

$$\dot{V}'''_{\rm s,gen} = 0, \qquad \dot{V}'''_{\rm f,gen} = 0 \tag{8}$$

and

$$\dot{V}'''_{\rm gen} = \dot{V}'''_{\rm s,gen} + \dot{V}'''_{\rm f,gen} = 0 \tag{9}$$

Assuming the velocity $\mathbf{w}$ to be the volume-average velocity, there follows for a binary system

$$\mathbf{j}_{\rm f}\bar{v}_{\rm f} + \mathbf{j}_{\rm s}\bar{v}_{\rm s} = 0 \tag{10}$$

and the substitutions of Eqs. (10) and (9) into Eq. (5) gives

$$\nabla\cdot\mathbf{w} = 0 \tag{11}$$

As the partial specific volumes $\bar{v}_i$ are assumed to be constant, they can be replaced by the corresponding specific volumes v_i.

SUPERFICIAL VELOCITIES

Superficial velocities of slurry, solids, and liquid are defined as follows. For the slurry, the superficial velocity is the same as the volume-average velocity.

$$\mathbf{W} = \mathbf{w} = \mathbf{W}_s + \mathbf{W}_f \tag{12}$$

Superficial velocities of solids and liquid are defined with a convective flux term and a "diffusional flux" term, in the same way as total fluxes are defined in diffusion. Hence, for solids one has

$$\mathbf{W}_s = (cv_s)\mathbf{w} + \mathbf{j}_s v_s \tag{13}$$

where $\mathbf{W}_s$ is the superficial velocity of solids relative to stationary coordinates, and $\mathbf{j}_s v_s$ is the "diffusional velocity" of solids relative to volume average velocity $\mathbf{w}$.

One has, thus, for the liquid

$$\mathbf{W}_f = (c_f v_f)\mathbf{w} + \mathbf{j}_f v_f \tag{14}$$

VELOCITIES

The true velocities are given by the following equations. For the slurry, one has

$$\mathbf{w} = \mathbf{W} \tag{15}$$

For the solids, the velocity is

$$\mathbf{u} = \mathbf{W}_s / cv_s \tag{16}$$

and for the liquid, the velocity in the interstices of the solids is

$$\mathbf{u}_f = \mathbf{W}_f / c_f v_f \tag{17}$$

Wallis (17) has also presented the different velocities in two-phase flow using volume-average velocity and a relative velocity.

DARCY'S EQUATION FOR LAMINAR FLOW

The well-known Darcy equation for laminar flow can be written as

$$\mathbf{u}_f - \mathbf{u} = \frac{1}{c_f v_f} \frac{K}{\eta} \nabla p_{f,f} \tag{18}$$

The pressure loss gradient $\nabla p_{f,f}$ is opposite in sign to the liquid pressure gradient ∇p_f for horizontal flow. The pressure loss gradient $\nabla p_{f,f}$ should be used in Eq. (18) and not the liquid pressure gradient ∇p_f, which is used by Stamatakis and Chi Tien (3) and Tiller et al. (4).

The permeability K is not constant for a compressible bed, but varies throughout the cake. It is usually presented as a function of consistency of solids c or compressible pressure p_s.

With Eqs. (16) and (17), Darcy's equation takes the form

$$\frac{\mathbf{W}_f + \mathbf{W}_s}{c_f v_f} - \frac{\mathbf{W}_s}{c_f v_f} - \frac{\mathbf{W}_s}{c v_s} = \frac{1}{c_f v_f}\frac{K}{\eta}\nabla p_{f,f} \tag{19}$$

Using Eq. (12) in Eq. (19), one has

$$\frac{\mathbf{w}}{c_f v_f} - \frac{\mathbf{W}_s}{c_f v_f c v_s} = \frac{1}{c_f v_f}\frac{K}{\eta}\nabla p_{f,f} \tag{20}$$

And with Eq. (16) we obtain Darcy's equation in the form

$$\mathbf{w} - \mathbf{u} = \frac{K}{\eta}\nabla p_{f,f} \tag{21}$$

Now the "diffusional mass flux" of solids can be expressed with Eqs. (13), (16), and (21):

$$\mathbf{j}_s = c(\mathbf{u} - \mathbf{w}) = -\frac{Kc}{\eta}\nabla p_{f,f} \tag{22}$$

In the above discussion inertial forces are neglected and gravitational forces are partly taken into account. Only the friction between particles and liquid is considered. The internal forces in the liquid due to viscosity are not considered, and wall friction is neglected.

CONSTITUTIVE EQUATION

The structural properties of a filter cake can be represented alternatively with porosity ϵ, solidosity ϵ_s, or consistency c. These variables can be related to each other.

Constitutive equations are used to correlate the structural bed properties with the compressive pressure acting on the solid particles. These equations are empirical, and different equations have been proposed.

Tiller et al. (4) suggested the following constitutive equation which is often utilized in the mathematical modeling of filtration:

$$\epsilon_s = \epsilon_s^0\left(1 + \frac{p_s}{p_A}\right)^{\beta} \tag{23}$$

Constants ϵ_s^0, p_A, and β in Eq. (23) are material specific.

According to Tiller et al. (4), this constitutive equation is generally valid for compressive pressures up to 5–10 atm. It is clearly seen from Eq. (23) that ϵ_s tends to infinity as $p_s \rightarrow \infty$. However, values of solidosity greater than unity have no physical meaning.

In the filtration of pulp suspensions, consistency c usually describes the structure of fiber beds. Fiber consistency is normally defined as weight of dry fibers per unit volume of suspension. The volume fraction of liquid, porosity, can be calculated from the fiber specific volume and the fiber consistency:

$$\epsilon = 1 - cv_s \tag{24}$$

Qviller's equation (18) is often used as a constitutive equation for fiber beds. In this equation, fiber consistency is correlated to compressive pressure with a power function.

$$c = Mp_s^N \tag{25}$$

In the following discussion, a general form of the constitutive equation is used

$$p_s = G(c) \tag{26}$$

PERMEABILITY

There are several correlations for the calculation of permeability. One of the most frequently used is obtained from Darcy's and Kozeny-Carman's equations.

$$K = \frac{(1 - cv_s)^3}{kS_0^2(cv_s)^2} \tag{27}$$

The Kozeny factor k in Eq. (27) depends on particle size, shape, and porosity. An average value of the Kozeny factor is 5 for roughly spherical particles in the porosity range between 0.3 and 0.5 (18). However, this average value is not applicable to beds formed of multisized particles.

For cellulose fibers the value of the Kozeny factor k is 5.55 for porosities less than 0.80 (19). For fiber structures with higher porosities, both Davies (20) and Carrol (21) proposed a correlation for the variation of Kozeny factor with porosity. Davies' correlation is applicable to porosities higher than 0.60 and Carrol's correlation for the whole porosity range. Carrol's correlation is of the form

$$k = 5.0 + e^{[14.0(cv_f - 0.80)]} \tag{28}$$

Tiller et al. (4) and Stamatakis and Chi Tien (3) correlated permeability to the compressive pressure in their studies with the following empirical equation:

$$K = \frac{K_0}{\left(1 + \frac{p_s}{p_A}\right)^{\delta}} \tag{29}$$

In Eq. (29), K_0 is the permeability at zero compressive stress. The constants K_0, p_A, and δ are material specific.

FORCE BALANCE

In a stationary bed, the total pressure at some arbitrary point is assumed to be constituted of liquid pressure p_f and compressive pressure on solids p_s.

$$\begin{aligned} p &= p_s + p_f \\ &= \int_0^{\mathbf{r}} \frac{1}{v_{sl}} \mathbf{g} \cdot d\mathbf{r} + p_0 \end{aligned} \tag{30}$$

The pressure gradients are

$$\nabla p = \nabla p_s + \nabla p_f = \mathbf{g}/v_{sl} \tag{31}$$

$$\nabla p_s = c v_s \left(\frac{1}{v_s} - \frac{1}{v_f}\right) \mathbf{g} \tag{32}$$

$$\nabla p_f = \mathbf{g}/v_f \tag{33}$$

The solid particles are assumed to lie on each other in the bed, and thus they are not contributing to the liquid pressure due to height.

Liquid flow is usually laminar in a nonstationary bed, and therefore inertial forces may be omitted. When wall friction is also neglected and particles are assumed to be in point contact with each other, only frictional forces between fluid and solid particles and gravitational forces need to be considered.

Thus we obtain Eqs. (32) and (33) in the following form for flow in a nonstationary bed:

$$\nabla p_s = c v_s \left(\frac{1}{v_s} - \frac{1}{v_f}\right) \mathbf{g} + \nabla p_{s,f} \tag{34}$$

$$\nabla p_f = \frac{\mathbf{g}}{v_f} - \nabla p_{f,f} \tag{35}$$

As an approximation, the gradient of the total pressure in Eq. (31) is assumed to be the same in the nonstationary bed as in a stationary bed with the same distribution of solids.

From the assumptions we obtain the relationship $\nabla p_{s,f} = \nabla p_{f,f}$, and therefore the liquid pressure loss gradient $\nabla p_{f,f}$ is

$$\nabla p_{f,f} = \nabla p_s - cv_s \left(\frac{1}{v_s} - \frac{1}{v_f}\right) \mathbf{g} \tag{36}$$

And when the last term in Eq. (36) is neglected, which is usually done in filtration, the force balance is

$$\nabla p_{f,f} = \nabla p_s \tag{37}$$

DIFFUSION EQUATIONS

The equation of the "diffusional mass flux" of solids Eq. (22) combined with the constitutive equation Eq. (26) and the force balance Eq. (37) gives

$$\mathbf{j}_s = -D(c)\nabla c \tag{38}$$

where

$$D(c) = \frac{Kc}{\eta}\frac{d[G(c)]}{dc}$$

Equation (38) is exactly the same equation as we have for diffusion mass transfer with a nonconstant volumetric diffusivity [$D(c) = ()\ m^2/s$]. In ordinary diffusion, this is called Fick's law.

The diffusivity of solids $D(c)$ may also be expressed in the following form using Eq. (27) for permeability, Carrol's correlation (Eq. 28) for the Kozeny factor, and Qviller's correlation (Eq. 25) for compressive pressure:

$$D(c) = \frac{(1 - cv_s)^3 c^{((1/N)-2)}}{\eta S_0^2 N M^{1/N} v_s^2 (5.0 + e^{14.0(0.2 - cv_s)})} \tag{39}$$

In Eq. (39), v_s is the volume of wet fibers per unit weight of dry fibers and the consistency c is the weight of dry fibers per unit volume.

Insertion of $\dot{V}'''_{s,gen} = 0$ and Eqs. (38) and (11) in Eq. (6) gives

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + \mathbf{w}\cdot\nabla c = \nabla\cdot[D(c)\nabla c] \tag{40}$$

This is the usual partial differential equation for diffusion with no reactions and no generation of volume.

If the solids contain liquid, which follows the solids, then Eqs. (31), (34), (35), (37), (38), and (40) (but not Eq. 39) hold with the following definitions:

ϵ_s volume fraction of wet solids
v_s volume of wet solids per unit weight of wet solids
c consistency, i.e., weight of wet solids per unit volume
$\mathbf{j}_s$ mass flux of wet solids

These equations also hold for ϵ_s and v_s as above and

c weight of dry solids per unit volume
$\mathbf{j}_s$ mass flux of dry solids

INCOMPRESSIBLE FILTER CAKES

The consistency c is often assumed to be constant in incompressible filter cakes, and then constitutive equations involving consistency cannot be used. Therefore, with incompressible cakes, we start with Eq. (21), the Darcy equation

$$\nabla p_{\mathrm{f,f}} = \frac{\eta(\mathbf{w} - \mathbf{u})}{K} \tag{41}$$

The boundary condition at the cake surface is obtained from the solids mass balance

$$\mathbf{w}_b = \frac{d\mathbf{r}_b}{dt} = \frac{c_F\mathbf{u} - c_c\mathbf{u}_c}{c_F - c_c} \tag{42}$$

where $\mathbf{w}_b$ and $\mathbf{r}_b$ are boundary velocity and radius vector, respectively. In Eq. (42), $\mathbf{u}_c$ is the velocity of particles in the cake at the cake surface and c_c is the cake consistency.

EXAMPLE

Cake filtration with variable hydrostatic pressure and sedimentation of solid particles is presented as an example. The filter cake is taken as incompressible. The influence of sedimentation on the filtration of incompressible filter cakes has also been studied by Theliander (22), Vadja and Törös (23, 24), and Bockstal et al. (25).

The filtration process in our example is presented schematically in Fig. 1. Filtration starts with slurry of consistency c_F, and the initial height of the slurry above the filter cloth is h_t. A constant gauge pressure of magnitude p_g is applied to the top of the slurry. Therefore, at the start of the

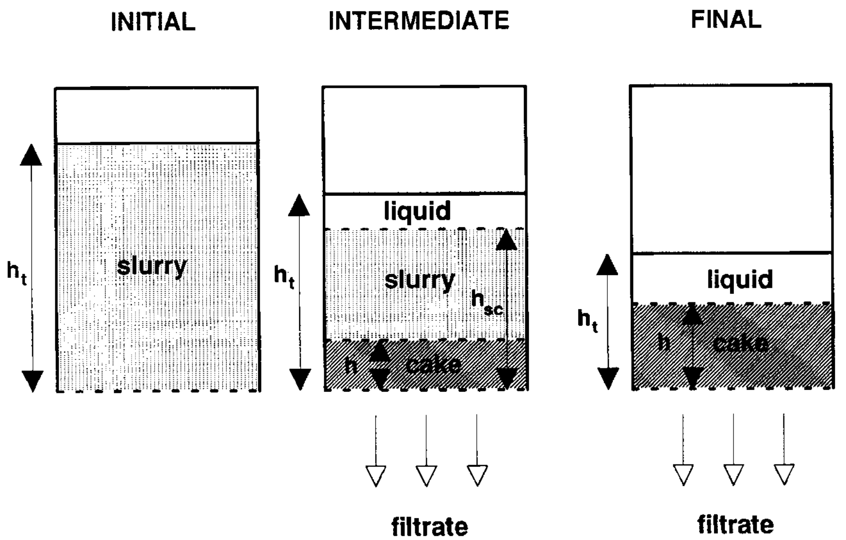

FIG. 1 The filtration process.

filtration the pressure difference across the filter cloth is the sum of the gauge pressure and the hydrostatic pressure of the slurry.

At an intermediate time, the height of the cake is h and the combined height of the cake and the slurry is h_{sc}. There is also a layer of clear liquid above the slurry, which is formed from the settling of the solid particles. The total height of solids and liquid above the filter cloth diminishes as the filtrate flows out of the filter chamber. The pressure drop across the cake and the filter cloth is the sum of the gauge pressure, the hydrostatic pressure of the slurry, and the hydrostatic pressure of the liquid.

The filter medium resistance is assumed to be constant during the filtration. The filtration process stops when all the solid particles are in the cake. This step is followed by the flow of the clear liquid through the cake, which is not included in these calculations.

Solution

The velocity of the particles in the cake is zero, as the cake in this example is incompressible. Therefore the solids mass balance for the one-dimensional system simplifies to

$$-w_b = \frac{dh}{dt} = \frac{c_F u}{c_c - c_F} = \frac{c_F(w + u_{st})}{c_c - c_F} \tag{43}$$

where u is the velocity of the particle in the slurry, and u_{st} is the hindered settling velocity of the particles, i.e., the velocity difference between the particles and the slurry.

The velocity of the sedimenting particles relative to the slurry is calculated in the same way as the hindered settling velocity is calculated in sedimentation. The solid particles are assumed to be spherical and of equal size. It is also assumed that the slurry concentration above the cake is constant. Thus, the settling velocity is constant.

Khan and Richardson (26) present the following correlation for the calculation of the terminal velocity u_t which applies to $\mathrm{Re_p} < 10^5$

$$\mathrm{Re_p} = \frac{\rho_f d u_t}{\eta} = (2.33\mathrm{Ga}^{0.018} - 1.53\mathrm{Ga}^{-0.016})^{13.3} \quad (44)$$

In Eq. (44), Ga is the dimensionless Galileo number:

$$\mathrm{Ga} = \frac{d^3 \rho_f(\rho_s - \rho_f) g}{\eta^2} \quad (45)$$

The hindered settling velocity u_{st} is calculated from the terminal settling velocity using the empirical equation

$$u_{st} = u - w = u_t(1 - c_F v_s)^n \quad (46)$$

The exponent n in Eq. (46) is calculated from the correlation presented by Khan and Richardson (27):

$$\frac{4.8 - n}{n - 2.4} = 0.043\mathrm{Ga}^{0.57} \quad (47)$$

We use Darcy's equation to calculate the flow of filtrate:

$$w = \frac{\Delta p_{f,f}}{\eta\left(\frac{h}{K} + R_m\right)} \quad (48)$$

where the pressure drop of fluid from friction expressed with the hydrostatic pressures is

$$\Delta p_{f,f} = p_g + \rho_{sl} g(h_{sc} - h) + \rho_f g(h_t - h_{sc} + h) \quad (49)$$

The pressure drop in Eq. (49) may often be approximated with the equation

$$\Delta p_{f,f} \approx p_g + \rho_{sl} g h_t \quad (50)$$

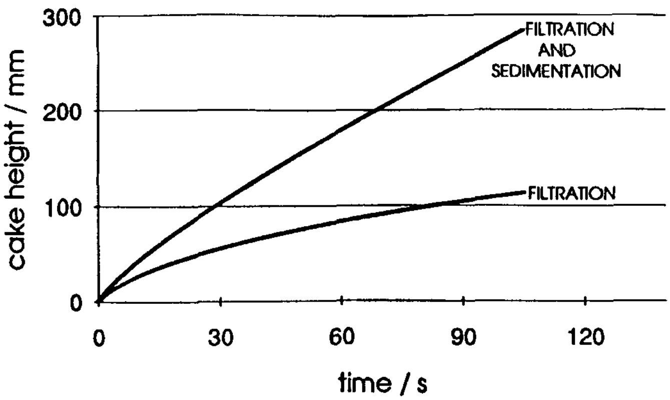

FIG. 2 Comparison between filtration and filtration with sedimentation.

The thickness of the clear liquid above the slurry is a function of the hindered settling velocity and of time:

$$h_t - h_{sc} = u_{st}t \tag{51}$$

The decrease of the total height in the filter chamber is proportional to the flow of filtrate:

$$dh_t/dt = -w \tag{52}$$

This system of five equations (Eqs. 43, 48, 49, 51, and 52) is solved with the method of finite differences. The results are shown in Figs. 2 and 3. In Fig. 2 we have compared the formation of the filter cake between

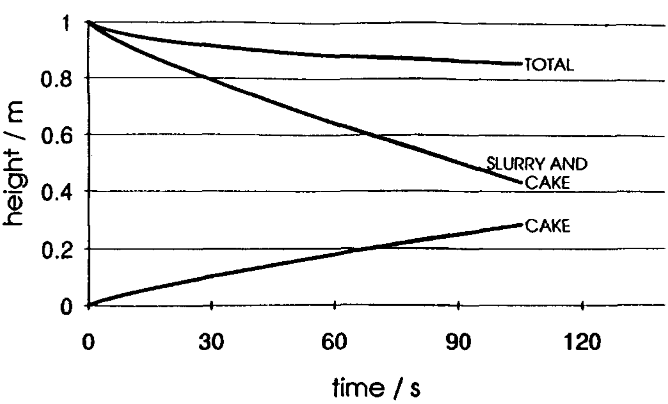

FIG. 3 Different heights in the filter chamber.

cake filtration with sedimentation with filtration where the effects of sedimentation have been neglected, and in Fig. 3 the different heights in the filter chamber are shown as a function of time. The following data were used in the calculations:

d = 150 μm
c_c = 1902 kg/m^3
c_F = 634 kg/m^3
p_g = 5 kPa
h_t = 1 m (at t = 0 seconds)
$K = 9.88 \times 10^{-12}$ m^2 (calculated from Eq. (27))
ρ_{sl} = 1434 kg/m^3
ρ_f = 1000 kg/m^3
ρ_s = 3170 kg/m^3
η = 1.005 mPa·s
$R_m = 2 \times 10^9$ m^{-1}

CONCLUSIONS

Equations describing the formation of filter cakes in cake filtration have been developed in this study. The general equations are applicable to three-dimensional cases, and both compressible and incompressible cakes are considered.

The filtration equations are derived to the same form as partial differential equations are in diffusion. This facilitates the solution of the differential equations as the mathematics of diffusion has been studied extensively.

The calculation procedure for an incompressible cake is presented in an example.

NOTATION

c	consistency of solids (kg/m^3)
c_c	cake consistency (kg/m^3)
c_i	consistency of component i (kg/m^3)
c_F	initial and feed consistency (kg/m^3)
d	diameter of particle (m)
$D(c)$	diffusivity of solids $\left[= \frac{Kc}{\eta} \frac{d[G(c)]}{dc}\right]$ (m^2/s)
$\mathbf{g}$	gravitation vector in downward direction (m/s^2)
Ga	dimensionless Galileo number
$G(c)$	constitutive function (Pa)

$\mathbf{j}_s$, $\mathbf{j}_f$ diffusional mass flux of solids and liquid with respect to average flow velocity $\mathbf{w}$ [kg/(m^{2}s)]
h cake thickness (m)
h_t total height of material in filter chamber (m)
h_{sc} thickness of cake and slurry in filter chamber (m)
k Kozeny factor
K permeability (m^2)
K_0 permeability of unstressed bed ($p_s = 0$) (m^2)
M compressibility constant (Eq. 25) [(kg/m^3)/Pa^N]
N compressibility constant (Eq. 25)
n empirical exponent in Eq. (47)
p total pressure (Pa)
p_0 constant total reference level pressure ($h = 0$) (Pa)
p_A empirical constant (Pa)
p_f liquid pressure (Pa)
p_s solids compressive pressure (Pa)
p_g gauge pressure (Pa)
$\nabla p_{f,f}$ liquid pressure loss gradient (Pa/m)
$\nabla p_{s,f}$ gradient of compressive solids stress due to friction (Pa/m)
$\Delta p_{f,f}$ pressure loss in liquid over cake and filter cloth (Pa)
r rate of generation of mass per unit volume [kg/(m^{3}s)]
$\mathbf{r}$ radius vector (m)
$\mathbf{r}_b$ boundary radius vector (m)
R_m filter medium resistance (m^{-1})
Re_p particle Reynolds number
S_0 specific surface (m^2/m^3)
t time (seconds)
$\mathbf{u}$ velocity of solids (m/s)
$\mathbf{u}_c$ velocity of solids in cake (m/s)
$\mathbf{u}_f$ velocity of liquid (m/s)
u_{st} hindered settling velocity of particles (m/s)
u_t terminal settling velocity of particles (m/s)
$\dot{V}'''_{gen}$ volume generation in total volume balance [m^3/(m^{3}s)]
$\dot{V}'''_{s,gen}$ volume generation in solids volume balance [m^3/(m^{3}s)]
$\dot{V}'''_{f,gen}$ volume generation in liquid volume balance [m^3/(m^{3}s)]
v specific volume (m^3/kg)
v_s, v_f specific volume of solids and liquid (m^3/kg)
$\bar{v}_s$, $\bar{v}_f$ partial volume of solids and liquid (m^3/kg)
$\mathbf{w}$ average (usually volume) velocity (m/s)
$\mathbf{w}_b$ boundary velocity at cake and slurry interface (m/s)
$\mathbf{W}$ superficial velocity of slurry (m/s)
$\mathbf{W}_s$ superficial velocity of solids (m/s)
$\mathbf{W}_f$ superficial velocity of liquid (m/s)

Greek Letters

β	compressibility coefficient
δ	compressibility coefficient
ϵ	porosity
ϵ_s	solidosity
ϵ_s^0	solidosity of unstressed bed ($p_s = 0$)
η	dynamic viscosity of liquid (Ns/m^2)
ρ	mass density (kg/m^3)

Subscripts

c	cake
f	liquid
i	component *i*
s	solid
sl	slurry

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